university of groningen

Exam Computer Assisted Problem Solving (CAPS)

April 4th 2018   14.00-17.00

This exam is 'closed book'. It is NOT allowed to use a book or lecture notes. Only the use of a simple pocket calculator is allowed. Programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, etc.).

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

1. Consider the equation $\sin(\frac{x}{4}) = \cos(\frac{x}{4})$, with for $x \in [2,4]$ the only exact solution $x = \pi$. To find the value of π, one could use an iterative method, with initial value $x_0 = 3$.

   (a) $\boxed{8}$ (1) Compute x_1 with Newton's method (one iteration), starting with $x_0 = 3$. Determine the most accurate ('the best') error estimate for x_1.
   (2) Compute two iterations with the Bisection method, with $I_0 = [2,4]$ as initial search interval (and hence $m_0 = 3$), without making use of the exact value of π. Show that only a few more iterations are needed for the same accuracy as in (1). So the Bisection method is comparably fast for this problem? Explain.

   (b) $\boxed{9}$ Someone uses the iterative method $x_{n+1} = x_n + \alpha\cos(\frac{x_n}{4}) - \alpha\sin(\frac{x_n}{4})$, with $x_0 = 3$. At first, the value $\alpha = 1$ is taken. In that case, the first 4 iterations are given by

x_0	x_1	x_2	x_3	x_4
3.00000000	3.05005011	3.08241246	3.10333506	3.11686095

   (1) Determine the theoretical error reduction factor and the convergence rate $\tilde{K}$. Compare these to the reduction rate of exact errors (use exact value of π).
   (2) Determine an error estimate for x_4 and compare it to the true error in x_4.
   (3) Calculate an improved solution for x_4 by means of Steffensen extrapolation.
   (4) For which α does the method converge? Determine the most optimal value of α.

   (c) $\boxed{8}$ Give a complete program (in pseudo code or Matlab-like language) that solves the problem using the Secant method, with an accuracy of tol=1E-6. Use an appropriate stopping criterion and start-up procedure. Your program should be as computationally efficient as possible.

2. Consider the integral
$$I_1 = \int_0^1 e^{\sqrt{x}}\, dx = 2$$

   (a) $\boxed{9}$ (1) Derive Simspon's from the Trapezoidal method (show/derive general formulas).
   (2) Approximate I_1 with Simpson's method on a grid with only 1 segment.
   (3) Is the global error theorem for Simpson useful for this integral? Explain.

   (b) $\boxed{8}$ Through the substitution $u = \sqrt{x}$ the integral is converted: $I_2 = \int_0^1 2\,u\,e^{u}\, du = 2$
   The results for the Trapezoidal(!) method, applied to both integrals, are given below.

n	$I_1(n)$	$I_2(n)$
4	1.97835496	2.04612896
8	1.99190779	2.01154821
16	1.99702730	2.00288805
32	1.99892076	2.00072208
64	1.99961132	2.00018052

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

P.T.O.

(1) Compute the q-factors for both I_1 and I_2 (see table). What can you conclude?
(2) Give an error estimate for $I_2(64)$ based on $I_2(n)$ values.
(3) Compute improved solutions (T_2) for $I_2(8)$ and $I_2(16)$ by means of extrapolation. Combine these extrapolations into a highly accurate approximation $T_3(16)$.
(4) Compare the result at (2) with the exact error in $T_3(16)$. How many intervals are required (powers of 2) for the Trapezoidal method to reach the accuracy of $T_3(16)$?

(c) $\boxed{8}$ Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Midpoint method (no extrapolation). Use an appropriate error estimate for the stopping criterion. Your program should be as computationally efficient as possible.

3. Consider on $[0,\ 6]$ the o.d.e. $\quad y'(x) = \dfrac{1}{y(x)} - x, \quad$ with boundary condition $y(0) = 1$.

(a) $\boxed{8}$ (1) Use explicit Euler to compute $y(x)$ at $x{=}1$ on a grid with $\Delta x{=}0.5$ (2 steps).
(2) Use Heun's method (RK2) to compute $y(x)$ at $x{=}0.5$ on a grid with $\Delta x{=}0.5$.
(3) Use the implicit(!) Euler method to compute $y(x)$ at $x{=}1$ on a grid with $\Delta x{=}1$.

(b) $\boxed{9}$ The explicit 3rd(!) order RK3 method is used on 2 coarse grids ($N{=}30, 60$ segments), and 3 finer grids. The table below shows solutions at a selection of x locations.

x_n	$N = 30$	$N = 60$	$N = 960$	$N = 1920$	$N = 3840$
4.8	0.20100403	0.21028199	0.2103099530989	0.2103099575739	0.2103099581125
5.0	0.90758040	0.20164462	0.2016690068014	0.2016690110630	0.2016690115744
5.2	0.26123492	0.19370630	0.1937273518644	0.1937273559404	0.1937273564279
5.4	0.42414186	0.18638281	0.1864008393445	0.1864008432572	0.1864008437237
5.6	0.53003059	0.17960342	0.1796187654317	0.1796187691996	0.1796187696472
5.8	0.31511378	0.17330818	0.1733211909195	0.1733211945577	0.1733211949884
6.0	1.78800005	0.16744589	0.1674569010218	0.1674569045435	0.1674569049589

(1) Compute the q-ratio for $x{=}5.0$ (finer grids). Conclusion?
(2) Derive (so not only give!) the appropriate extrapolation formula. Compute an improvement for the solution at $x = 5.0$ (extrapolation).
(3) Which N is sufficient (roughly) for an accuracy of 5.0E-4 on the full $[0,\ 6]$?
(4) Is there a stability limit visible? Explain.

(c) $\boxed{8}$ Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Heun method (without extrapolation). Use an appropriate error estimate for the stopping criterion.

4. Consider on $[0,\ 10]$ the diff. eqn. $\quad y''(x) + \alpha\, y'(x) + \frac{2}{10}\, y(x) = \cos(\frac{\pi}{2}\, x)$,
with boundary conditions (BC) $y'(0) = 3$ and $y(10) = 0$.

(a) $\boxed{6}$ Take $\alpha = 0$, such that $y'(x)$ is out of the diff. eqn. (apart from the left BC). Give the matrix-vector system, when the problem is solved on a grid with $N = 5$ segments by means of the matrix method, using the [1 -2 1]-formula for $y''(x)$.

(b) $\boxed{9}$ The term $y'(x)$ can be approximated at x_i by means of $y'(x_i) = \dfrac{y_i - y_{i-1}}{\Delta x}$.
(1) Which modifications do you have to make to the system in (a) when $y'(x)$ is treated in this way, if $\alpha = 1$ (instead of $\alpha = 0$)?
(2) Will it take longer to solve the system with TDMA in this case? Explain.
(3) Take a grid with $N = 2$ segments (1 interior point) and solve the system ($\alpha = 1$).
(4) On a grid with $N = 100$, will the solution for $\alpha = 1$ be more accurate than for $\alpha = 0$? Explain.

Total: $\boxed{100}$